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THE GROWTH RATE OF VAPOUR BUBBLES IN SUPERHEATED PURE LIQUIDS AND BINARY MIXTURES

PART I: THEORY

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Abstract—A survey is given of the theories concerning spherically symmetric growth of free bubbles in initially uniformly superheated pure liquids. The physical basis of the Bošnjaković model is developed from experiments by Jakob and Fritz, giving the temperature distribution in boiling liquids, and by Heidrich and Prüger, showing that thermodynamic equilibrium exists at the vapour–liquid interface during stationary evaporation of superheated liquids without ebullition.

According to Forster and Zuber, and Plesset and Zwick, bubble growth in a superheated pure liquid following from Rayleigh's dynamic equation of isothermal motion [$R \cong (2\Delta p/3\rho_1)^{\frac{1}{2}} t$ for an expanding spherical cavity with a constant excess pressure Δp] is slowed down by heat diffusion towards the bubble boundary to satisfy the latent heat requirement of evaporation. For asymptotic growth ($R_p \cong C_{1,p}\Delta\theta_0 t^{\frac{1}{2}}$), bubble dynamics and the influence of viscosity and surface tension are neglible since $\Delta p \to 0$ as $t \to \infty$ (isobaric growth). Thermodynamic equilibrium at the bubble boundary follows from the extended Rayleigh equation in accordance with Prüger's results.

In superheated binary mixtures, bubble growth is further decreased due to the analogous mass diffusion of the more volatile component according to van Wijk, Vos and van Stralen, Scriven, Bruijn, van Stralen and Skinner and Bankoff. Van Stralen's modification shows the physical equivalence of the various theories. The dew temperature of the vapour is increased with an amount ΔT with respect to the boiling temperature of the original liquid. As a consequence, the occurrence of a minimal bubble growth rate (corresponding with a maximal $\Delta T/G_d$) is predicted at a certain low concentration of the more volatile component. This leads to the "broiling paradox", which can be explained by van Stralen's "relaxation microlayer" theory.

NOMENCLATURE

a,	$= k/\rho_1 c$, liquid thermal diffusivity [m^2/s];
a_1,	horizontal semiaxis of ellipsoidal vapour bubble [m];
a_2,	vertical semiaxis of ellipsoidal vapour bubble [m];
A,	area of vapour–liquid interface [m^2];
A_w,	area of surface of platinum heating wire [m^2];
A_1,	$= 2(\rho_2 l/\rho_1 c)\beta^3 \exp(\beta^2 + 2\varepsilon\beta^2)$, constant of integration [degC];
A_2,	$= 4(\rho_2 l/\rho_1 c)\beta^2 \exp \beta^2$, constant of integration [degC];
$A(t)$,	amplitude of bubble vibrations [m];
b,	dimensionless bubble growth parameter during adherence;
B_1,	$= 2\sigma/\rho_1 R_0 \Delta\theta_0$, coefficient in extended Rayleigh equation [m^2/s^2 degC];
B_2,	$= (2\sigma/\rho_1 R_0^3)^{\frac{1}{2}}$, coefficient [1/s];
B_3,	$\cong 2/\pi\kappa$, dimensionless constant;
c,	liquid specific heat at constant pressure [J/kg degC];

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C_1,	$= C_2/\Delta\theta_0$, bubble growth constant for relatively large liquid superheatings [$\mathrm{m/s^{\frac{1}{2}}}$ degC];
C_1^*,	$= C_2/(\Delta\theta_0)^{0\cdot50}$, bubble growth constant for very small liquid superheatings [$\mathrm{m/s^{\frac{1}{2}}}$ $(\mathrm{degC})^{\frac{1}{2}}$];
C_1^{**},	$= C_2/(\Delta\theta_0)^{0\cdot75}$, bubble growth constant for moderate liquid superheatings [$\mathrm{m/s^{\frac{1}{2}}}$ $(\mathrm{degC})^{\frac{3}{4}}$];
C_2,	$= R/t^{\frac{1}{2}}$, bubble growth factor [$\mathrm{m/s^{\frac{1}{2}}}$];
d,	thickness of thermal liquid boundary layer, or of disturbed region (in nucleate boiling) at liquid-level surface [m];
d_b,	thickness of conduction layer surrounding bubble boundary [m];
D,	mass diffusivity of more volatile component in less volatile component [$\mathrm{m^2/s}$];
D_w,	diameter of heating wire [m];
E,	effective potential drop across platinum heating wire [V];
f,	$= (\omega_0^2 - \omega^2)^{\frac{1}{2}}$, viscous damping coefficient for bubble oscillations [1/s];
G_d,	vaporized mass diffusion fraction for individual bubble;
G_h^*,	vaporized heat diffusion fraction for individual bubble;
h_b,	$= q_b/\Delta\theta_0$ = coefficient of heat transfer to free or released vapour bubble [$\mathrm{W/m^2}$ degC];
I,	effective electrical current strength through platinum heating wire [A];
Ja,	$= (\rho_1 c/\rho_2 l)\Delta\theta_0$, dimensionless Jakob number for pure liquids;
Ja_0,	$= (\rho_1 c/\rho_2 l)\ \Delta\tau = (C_{1,m}/C_{1,p})\ Ja$, dimensionless modified Jakob number for binary mixtures;
k,	liquid thermal conductivity [W/m degC];
K,	$= y/x$, equilibrium constant of more volatile component in binary mixture (ratio of mass fractions);
l,	latent heat of vaporization [J/kg];
L_w,	length of heating wire [m];
m,	parameter for modes of vibration, $m = 2$ for fundamental harmonic;
n,	exponent of $\Delta\theta_0$ in bubble growth equation for moderate liquid superheatings, $0\cdot50 \leqslant n \leqslant 1\cdot00$;
n^*,	= exponent of time in bubble growth equation $R \sim t^{n^*}$;
p,	$= (R/R_0)^3$, dimensionless variable replacing R;
p,	$= \mathrm{d}p/\mathrm{d}u$;
p_0,	$= p_v(T) = p_v(T + \Delta\theta_0) - 2\sigma/R_0$, ambient pressure, or liquid pressure at great distance from bubble [$\mathrm{kg/m\ s^2}$];
p_v,	saturated vapour pressure [$\mathrm{kg/m\ s^2}$];
$p_v(T)$,	saturated vapour pressure at boiling temperature T [$\mathrm{kg/m\ s^2}$];
$p_v(T + \Delta\theta_0)$,	$= p_v(T_{R_0}) = p_v(T_0)$, initial vapour pressure in equilibrium bubble [$\mathrm{kg/m\ s^2}$];
$p_v(T + \Delta\theta_0) - (2\sigma/R_0)$,	initial pressure in liquid at boundary of equilibrium bubble [$\mathrm{kg/m\ s^2}$];
$p_v(T_R)$,	instantaneous saturated vapour pressure in bubble [$\mathrm{kg/m\ s^2}$];
$p(R)$,	$= p_v(T_R) - (2\sigma/R)$, instantaneous pressure on liquid side of bubble boundary [$\mathrm{kg/m\ s^2}$];
$p(R) - p_0$,	$= \Delta p - 2\sigma/R = (2\sigma/R_0\Delta\theta_0)\Delta T_R^* + (2\sigma/R_0)(1 - R_0/R)$ = instantaneous driving pressure difference, or instantaneous pressure difference between liquid at bubble boundary and at great distance from bubble [$\mathrm{kg/m\ s^2}$];
$\Delta p(t)$,	$= p(R) - Hp_0 + 2\sigma/R = p_v(T_R) - p_0 = p_v(T_R) - p_v(T) = (\rho_2 l/T)\ \Delta T_R(t)$, instantan-

	eous difference in pressure on vapour side of bubble boundary and in liquid at great distance from bubble; $\Delta p(t) \to 0$ as $t \to \infty$ $[\mathrm{kg/m\,s^2}]$;
q,	rate of heat flow through unit area of surface, or heat flux density $[\mathrm{W/m^2}]$;
Q,	strength per unit volume and time of constant heat source starting initial growth from equilibrium bubble $[\mathrm{kg/ms^3}]$;
r,	distance to bubble centre [m];
R,	instantaneous (equivalent) bubble radius [m];
$\dot{R}$,	$= \mathrm{d}R/\mathrm{d}t$, instantaneous radial velocity of bubble boundary, or bubble growth rate [m/s];
$\ddot{R}$	$= \mathrm{d}^2R/\mathrm{d}t^2$, instantaneous radial acceleration of bubble boundary $[\mathrm{m/s^2}]$;
R_0,	$= R(0) = 2\sigma T/\rho_2 l\Delta\theta_0$, equilibrium radius of bubble in liquid of uniform superheating $\Delta\theta_0$, $R_0 = 2\sigma T/\rho_2 l\theta_0$ at superheating θ_0 of heating surface [m];
R_1,	$= R(t_1)$, bubble radius at the instant t_1 of breaking away from heating surface (exception: Fig. 2 of Part I) [m];
s,	$= r/2(at)^{\frac{1}{2}}$, dimensionless variable relating distance to penetration depth for heat conduction;
t,	time elapsed since initial formation of bubble [s];
t_1,	instant, at which bubble is breaking away from heating surface [s];
t',	$= z_1 t$, time of moving instantaneous heat sink at distance $R(t')$ from bubble centre [s];
T,	absolute boiling point, or saturation temperature at ambient pressure p_0 [°K];
$T(r, t)$,	absolute temperature [°K];
T_0,	$= T + \Delta\theta_0$, initially uniform temperature of superheated liquid; $T_0 = T(x_0) + \Delta\theta_0$ in binary liquid mixture [°K];
T_R,	$= T(R, t)$, instantaneous absolute temperature at bubble boundary, or instantaneous uniform temperature of vapour in bubble; $T_R \to T$ as $t \to \infty$ [°K];
ΔT_R,	$= T_R - T$, instantaneous superheating of liquid at bubble boundary, and of vapour in bubble; $\Delta T_R \to 0$ as $r \to \infty$ [degC];
ΔT_R^*,	$= \Delta T_R - \Delta\theta_0 = -(T_0 - T_R)$, instantaneous temperature difference between liquid at bubble boundary and at great distance; $\Delta T_R^* < 0$, $\Delta T_R \to -\Delta\theta_0$ as $t \to \infty$ [degC];
$T(x)$,	absolute boiling temperature of liquid at bubble boundary in binary liquid mixture [°K];
$T(x_0)$,	absolute boiling temperature of original liquid in binary liquid mixture, or boiling point of bulk liquid in mixture [°K];
$T(y)$,	$= T(x)$, absolute dew temperature of saturated vapour in binary mixture [°K];
ΔT,	$= T(x) - T(x_0) = T(y) - T(x_0)$, temperature difference between dew temperature of vapour in bubbles and boiling temperature of bulk liquid in binary mixture, or increase in temperature of liquid at bubble boundary with respect to original liquid; $\Delta T = 0$ for pure liquids and for azeotropic mixture [degC];
u,	$= (B_2/R_0^4)\int_0^t R^4(z_3)\,\mathrm{d}z_3$, dimensionless variable replacing t;
v,	$= v(r, t) = \varepsilon(R^2/r^2)\dot{R}$, instantaneous liquid velocity at distance $r > R$ from bubble centre [m/s];
$v(R, t)$,	$= \varepsilon\dot{R}$, instantaneous velocity of liquid immediately adjacent to bubble boundary [m/s];

V,	equivalent bubble volume [m^3];
x,	$= x_0/\{1 + (K - 1)G_d\}$, mass fraction of more volatile component in liquid at bubble boundary in binary liquid mixture;
$x\{R_m(t)\}$,	instantaneous mass fraction of more volatile component in liquid at bubble boundary in binary liquid mixture;
x_0,	mass fraction of more volatile component in originally homogeneous liquid or bulk liquid in binary mixture;
y,	$= Kx$, mass fraction of more volatile component in vapour of binary mixture;
$z_1, \ldots, z_5$,	integration variables.

Greek symbols

α,	evaporation rate per unit area [kg/m^2 s];
β,	$= R/2(at)^{\frac{1}{2}} = C_2/2a^{\frac{1}{2}}$, dimensionless bubble growth coefficient;
ε,	$= 1 - \rho_2/\rho_1$, constant, taking effect of radial convection on bubble growth into account;
ε_0,	elevation of boiling point in bulk liquid in binary mixture with nonvolatile component [degC];
ε_1,	elevation of boiling point in liquid at bubble boundary in binary mixture with nonvolatile component [degC];
$\Delta\varepsilon$,	$= \varepsilon_1 - \varepsilon_0$, increase in boiling point at bubble boundary with respect to original liquid in binary mixture with nonvolatile component [degC];
η,	liquid dynamic viscosity [kg/m s];
θ_0,	$= T_w - T$, temperature difference between heating surface and the bulk of boiling liquid, or superheating of platinum wire [degC];
$\Delta\theta_0$,	initially uniform liquid superheating above saturation temperature, or liquid superheating at great distance from bubble, or superheating of bulk liquid [degC];
κ,	$= (B_1\rho_2 l/3B_2 kR_0)(a/\pi B_2)^{\frac{1}{2}}$, dimensionless coefficient in transformed extended Rayleigh equation;
μ,	$= (a/D)^{\frac{1}{2}}$, dimensionless constant relating heat and mass diffusion;
ν,	$= \eta/\rho_1$, liquid kinematic viscosity [m^2/s];
ξ,	$= z_4/u$, substitution factor for transformation of integral in bubble growth equation;
ρ_1,	liquid density [kg/m^3];
ρ_2,	saturated vapour density [kg/m^3];
σ,	surface tension constant [kg/s^2];
τ,	period of damped bubble oscillations [s];
τ_0,	period of corresponding undamped bubble oscillations [s];
$\Delta\tau$,	$= \Delta\theta_0 - \Delta T$, "effective" liquid superheating for bubble growth in binary mixture [degC], $\Delta\tau = \Delta\theta_0$ for pure liquids and for azeotropic mixture, $\Delta\tau = \Delta\theta_0 - \Delta\varepsilon$ for mixture with nonvolatile component [degC];
Φ,	rate of heat flow [W];
$\Phi(\varepsilon, \beta)$,	$= (\rho_1 c/\rho_2 l)\ \Delta\tau$, dimensionless heat diffusion function for binary liquid mixture, denoted by $\varphi(\varepsilon, \beta)$ in [22];
$\Phi(\varepsilon, \mu\beta)$,	$= \rho_1(x_0 - x)/\rho_2(y - x) = (\rho_1/\rho_2)G_d$, dimensionless mass diffusion function for binary liquid mixture, denoted by $\varphi(\varepsilon, \mu\beta)$ in [22];
ω,	angular frequency of damped bubble oscillations [1/s];

ω_0,	angular frequency of corresponding undamped bubble oscillations [1/s].

Subscripts

b,	value for individual bubble;
d,	value for mass diffusion;
h,	value for heat diffusion;
m,	value in binary mixture;
max,	value for peak flux conditions;
p,	value in less volatile component (omitted in Section 1 of Part I);
w,	value for heating surface.

Numerical values for water at atmospheric boiling point

a, $= 16{\cdot}9 \times 10^{-8}\ \mathrm{m^2/s}$;
c, $= 4216\ J/\mathrm{kg\ degC}$;
$C_{1,p}$, $= 22 \times 10^{-4}\ \mathrm{m/s^{\frac{1}{2}}\ degC}$ (Forster–Zuber);
$24 \times 10^{-4}\ \mathrm{m/s^{\frac{1}{2}}\ degC}$ (Plesset–Zwick and Scriven for relatively large $\Delta\theta_0$);
$32 \times 10^{-4}\ \mathrm{m/s^{\frac{1}{2}}\ degC}$ (Scriven and van Stralen, experimental, for small $\Delta\theta_0 = 0{\cdot}11 - 0{\cdot}36$ degC);
Ja, $= 2{\cdot}995\ \Delta\theta_0$;
k, $= 0{\cdot}6825\ \mathrm{W/m\ degC}$;
l, $= 22{\cdot}56 \times 10^5\ \mathrm{J/kg}$;
R_0, $= 3{\cdot}3 \times 10^{-5}/\Delta\theta_0\ \mathrm{m}$;
β, $= 1220\ C_2$;
ε, $= 0{\cdot}9994$;
ρ_1, $= 958{\cdot}4\ \mathrm{kg/m^3}$;
ρ_2, $= 0{\cdot}598\ \mathrm{kg/m^3}$;
$\rho_2 l/\rho_1 c$, $= 0{\cdot}3339$ degC;
σ, $= 0{\cdot}0587\ \mathrm{kg/s^2}$.

Numerical values for water–methylethylketone ($x_0 = 4{\cdot}1 \times 10^{-2}$) at atmospheric boiling point (362°K)

$C_{1,m}$, $= 6 \times 10^{-4}\ \mathrm{m/s^{\frac{1}{2}}\ degC}$;
ν, $= 3{\cdot}3 \times 10^{-7}\ \mathrm{m^2/s}$;
σ, $= 0{\cdot}0414\ \mathrm{kg/s^2}$.

Numerical values for water–1-butanol ($x_0 = 1{\cdot}5 \times 10^{-2}$ and $6{\cdot}0 \times 10^{-2}$) at atmospheric boiling point (average value: 370°K)

$C_{1,m}$, $= 18 \times 10^{-4}\ \mathrm{m/s^{\frac{1}{2}}\ degC}$ at $\Delta\theta_0 = 0{\cdot}14$ degC for $x_0 = 1{\cdot}5 \times 10^{-2}$;
$21 \times 10^{-4}\ \mathrm{m/s^{\frac{1}{2}}\ degC}$ at $\Delta\theta_0 = 0{\cdot}31$ degC for $x_0 = 6{\cdot}0 \times 10^{-2}$;
D, $= 9{\cdot}9 \times 10^{-10}\ \mathrm{m^2/s}$ for 1-butanol in water;
μ, $= 13{\cdot}1$ for 1-butanol in water.

Numerical values for ethanol at atmospheric boiling point (351°K)

$C_{1,p}$, $= 8 \times 10^{-4}\ \mathrm{m/s^{\frac{1}{2}}\ degC}$ (Forster–Zuber);
$9 \times 10^{-4}\ \mathrm{m/s^{\frac{1}{2}}\ degC}$ (Plesset–Zwick, Scriven).

1. PURE LIQUIDS

1.1. *Introduction*

In 1930 **Bošnjaković** [1–4] calculated the heat flux density to a growing spherical vapour bubble ascending in a boiling pure liquid on account of the following hypotheses:

(i) The required latent heat of vaporization is supplied to the bubble boundary as a consequence of a slight superheating of the entire bulk liquid.
(ii) After a short initial stage, a vapour bubble is surrounded continuously by a thin boundary layer, through which heat transmission is due to conduction only.
(iii) Thermodynamic equilibrium is assumed at the bubble boundary.

The importance of hydrodynamics for bubble growth is thus neglected. Strictly speaking, this applies only to the asymptotic behaviour (cf. Sections 1.6–1.9). Anyhow, more exact recent theories have the same physical background.

1.2. *The superheating of the bulk of a boiling liquid*

1.2.1. *Heat transfer with evaporation at the heating surface (region of nucleate boiling of a boiling curve).* Bošnjaković's first hypothesis is justified by results of experiments carried out by Jakob and Fritz [2–5] for water boiling at atmospheric pressure. A boundary layer adhering to the metal heating plate was superheated considerably, and the entire remaining mass of liquid was superheated uniformly to approximately $\Delta\theta_0 = 0{\cdot}4$ degC for a smooth surface and to $\Delta\theta_0 = 0{\cdot}2$ degC, for a rough surface (Fig. 1), which shows a larger number of active nuclei generating vapour bubbles. The dew point of the vapour, which is developed at the water-level surface, equals the saturation temperature at the ambient pressure.

The superheating $\Delta\theta_0$ of the bulk liquid is independent of the heat flux density of the electrically heated bottom plate of the boiling vessel, at least for moderate values (Fig. 1).

1.2.2. *Heat transfer without evaporation at the heating surface (convection region of a boiling curve).* Heidrich [6] and Prüger [7] measured the temperature (in the absence of vapour bubbles) of superheated boiling water, and of water and carbontetrachloride, respectively. A glass boiling vessel was isolated thermally in

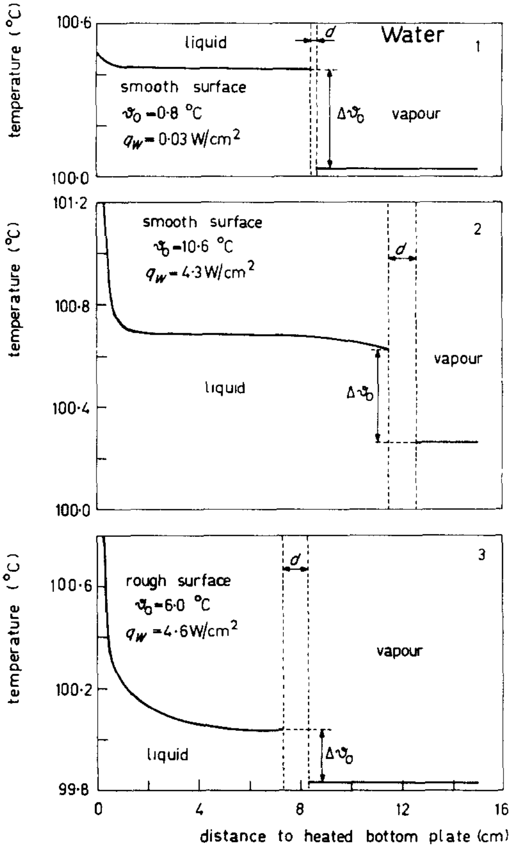

FIG. 1. *Water.* Temperature distribution above horizontal heating surface in water boiling approximately at atmospheric pressure (region of nucleate boiling), according to Jakob and Fritz. $\Delta\theta_0$ denotes the superheating of bulk liquid and d the thickness of the region disturbed by vapour bubbles at the water-level surface.

$\Delta\theta_0$ is uniform with exception of a boundary layer adhering to the heated bottom plate.

$\Delta\theta_0$ and $\boldsymbol{\Theta}_0$ (superheating of bottom plate) decrease at constant q_w with increasing roughness of heating surface (cf. 2 and 3) as the density of active nuclei generating vapour bubbles increases more rapidly.

$\Delta\theta_0$ is approximately independent of q_w for moderate values in the region of nucleate boiling (cf. 1 and 2). The dew temperature of vapour is the saturation temperature at ambient pressure.

order to minimize heat losses to the surroundings.

The liquid was superheated above the atmospheric boiling point by means of a (relatively small) heat flux density in the region of convection, i.e. without generation of vapour bubbles. Consequently, the temperature in the undisturbed boundary layer at the liquid-level surface could also be determined now. The heat flux density was calculated from the observed evaporation rate. Prüger measured the temperature drop in the liquid boundary layer at the liquid-level surface by means of a very small thermocouple (thickness of the wires 3×10^{-5} m, of the horizontal hot junction 4×10^{-5} m).

The results of this investigation are (Fig. 2):

(i) Liquid and vapour temperatures are approximately uniform with the exception of the liquid boundary layer.

(ii) The superheating in this boundary layer increases linearly with the distance to the liquid-level surface, whence heat transmission occurs by conduction only (justification of Bošnjaković's second hypothesis). This is analogous to the temperature drop in the equivalent conduction layer at the heating surface.

The thickness of the boundary layer amounts to $d = 3{\cdot}2 \times 10^{-4}$ m for carbontetrachloride at an evaporation rate per unit area of surface $\alpha = 8{\cdot}2 \times 10^{-3}$ kg/m^2 s (corresponding with a heat flux density of

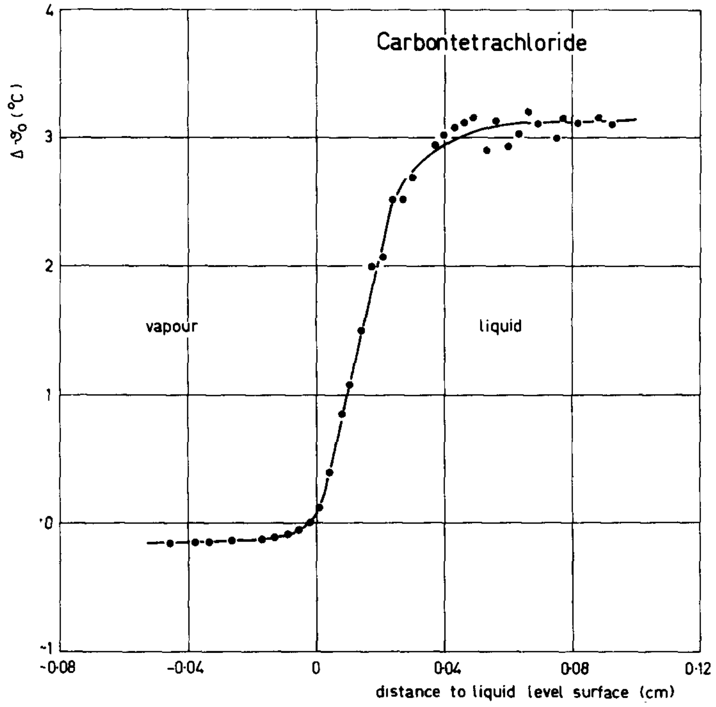

FIG. 2. *Carbontetrachloride.* Vaporization at atmospheric pressure without vapour formation at heating surface (region of convection of a boiling curve), according to Prüger.

The evaporation rate at the liquid-level surface amounts to $8{\cdot}2 \times 10^{-3}$ kg/m^{2}s. The temperature distribution in the liquid boundary layer is linear and is hence due to heat transmission by conduction only. Thermodynamic equilibrium occurs at the liquid–vapour interface.

The dew point of vapour is the saturation temperature, the liquid is superheated, vapour and liquid temperatures outside the liquid boundary layer are uniform.

$1{\cdot}85 \times 10^4$ W/m^2); $d = 6{\cdot}6 \times 10^{-4}$ m for water at $\alpha = 2{\cdot}0 \times 10^{-3}$ kg/m^2 s ($0{\cdot}46 \times 10^4$ W/m^2).

(iii) A practically negligible temperature jump occurs at the liquid–vapour boundary, viz. of 10^{-3} degC for carbontetrachloride and of 4×10^{-2} degC for water (justification of Bošnjaković's third hypothesis).

Apparently, this small temperature jump, which is connected to the condensation coefficient in the kinetic theory, suffices to maintain the vaporization process. This coefficient is of the order of magnitude of 10^{-2} for water and approaches the value 1 for nonpolar liquids as carbontetrachloride.

1.3. *The Bošnjakovič theory*

Evaporation at the liquid–vapour interface occurs on account of supply of heat from the superheated liquid by conduction through the boundary layer, i.e. the heat flux density equals:

$$q = \frac{\Phi}{A} = \alpha l = k\frac{\Delta\theta_0}{d}. \tag{1}$$

One calculates from equation (1) a superheating $\Delta\theta_0 = 3{\cdot}2$ degC for carbontetrachloride, since $k = 1{\cdot}59 \times 10^{-2}$ W/mdegC at the atmospheric boiling point $T = 349{\cdot}8$°K, by substitution of Prüger's experimental values for d and α, of Section 1.2.2. This is in good agreement with the actually measured value (3·3 degC, in Fig. 2).

One has for a spherical vapour bubble with $A_b = 4\pi R^2$:

$$\alpha_b = \rho_2 \dot{R}. \tag{2}$$

Hence:

$$q_b = \rho_2 l \dot{R} = k\left(\frac{\partial T}{\partial r}\right)_{r=R} = k\frac{\Delta\theta_0}{d_b}. \tag{3}$$

Equation (3) results in a bubble growth equation by taking simply $d_b = (at)^{\frac{1}{2}} = (kt/\rho_1 c)^{\frac{1}{2}}$, i.e. proportional to the expression valid in case of transient heat conduction for a homogeneous semi-infinite body with a plane boundary. It follows that

$$R(t) = 2\frac{\rho_1 c}{\rho_2 l}\Delta\theta_0 (at)^{\frac{1}{2}} = 2\,Ja(at)^{\frac{1}{2}}$$
$$= C_1 \Delta\theta_0 t^{\frac{1}{2}} = C_2 t^{\frac{1}{2}}, \tag{4}$$

where the Jakob number is defined by

$$Ja = \frac{\rho_1 c}{\rho_2 l}\Delta\theta_0. \tag{5}$$

$R(0) = R_0$ has been neglected, cf. Section 1.9. Note that $R \sim t^{\frac{1}{2}}$, which is characteristic for diffusion in contradiction to $R \sim t$ in equation (8).

The growth factor C_2 determines actual growth rates of vapour bubbles in boiling vessels. The coefficient C_1 is more suitable for comparing experimental results with theoretical predictions due to its independence of $\Delta\theta_0$, at least for asymptotic growth. Predictions of equation (4) differ only slightly from those of the similar, more exact equations (25) and (49) by Forster and Zuber [8] and by Plesset and Zwick [9], respectively.

It may be worth reporting, that (1) the thickness d_b of the boundary layer surrounding a growing vapour bubble is assumed to be independent of $\Delta\theta_0$; (2) d_b increases linearly with $t^{\frac{1}{2}}$, i.e. the ratio R/d_b should be independent of t and proportional to $\Delta\theta_0$.

1.4. *The Rayleigh equation of motion*

A spherical cavity in a non-viscous incompressible liquid with constant excess pressure

$$\Delta p = p(R) - p_0 \tag{6}$$

expands from an initial radius R_0 to R. The energy equation yields in combination with continuity [10]:

$$\tfrac{1}{2}\rho_1 \int_R^\infty v^2(r) 4\pi r^2\,\mathrm{d}r = 2\pi\rho_1 \int_R^\infty \left(\frac{R^2}{r^2}\dot{R}\right)^2 r^2\,\mathrm{d}r$$
$$= 2\pi\rho_1 R^3 \dot{R}^2 = \frac{4\pi}{3}\Delta p(R^3 - R_0^3). \tag{7}$$

Hence the cavity radius approaches a linear increase with time for $R \to \infty$ as then

$$R \cong \left(\frac{2\Delta p}{3\rho_1}\right)^{\frac{1}{2}} t. \tag{8}$$

Differentiation of equation (7) with respect to time and dividing by $R^2\dot{R}$ yields Rayleigh's equation of motion:

$$R\ddot{R} + \tfrac{3}{2}\dot{R}^2 = \frac{\Delta p}{\rho_1} \tag{9}$$

or

$$\frac{1}{2R^2\dot{R}}\frac{\mathrm{d}}{\mathrm{d}t}(R^3\dot{R}^2) = \frac{\Delta p}{\rho_1}. \tag{10}$$

For actual, viscous liquids, a dissipation term $4\eta\dot{R}/\rho_1 R$ has to be added to the left-hand side of equations (9) and (10). This term is neglected in the following as the viscosity of many liquids is small. Moreover, in case of asymptotic bubble growth, $\dot{R}/R \sim 1(t \to 0$ as $t \to \infty$, cf. equation (24).

1.5. *The extended Rayleigh equation*

It is shown here, that the expressions, which are used by Forster and Zuber [8] on one hand and by Plesset and Zwick [9] on the other hand, and which may seem to be different at first sight, are identical.

Forster and Zuber write for the extended Rayleigh equation:

$$\rho_1(R\ddot{R} + \tfrac{3}{2}\dot{R}^2) = \Delta p - \frac{2\sigma}{R}, \tag{11}$$

because the liquid is inserting a pressure $2\sigma/R$ due to surface tension inside the cavity. Thomson's correction—$(2\sigma/R)\rho_2/(\rho_1 - \rho_2) \ll 2\sigma/R$ at low pressures—for the decrease of vapour pressure at a concave liquid surface has been neglected in the right-hand side of equation (11).

One has thus in combination with Section 1.6 (Fig. 3):

$$\begin{aligned}\rho_1(R\ddot{R} + \tfrac{3}{2}\dot{R}^2) &= \{p_v(T_R) - p_v(T)\} \\ &\quad - \frac{2\sigma}{R} = \frac{2\sigma}{R_0}\frac{T_R - T}{\Delta\theta_0} - \frac{2\sigma}{R} \\ &= \frac{2\sigma}{R_0}\frac{T_R - T_0}{\Delta\theta_0} + \frac{2\sigma}{R_0}\left(1 - \frac{R_0}{R}\right) \\ &= \frac{2\sigma}{R_0}\frac{\Delta T_R^*}{\Delta\theta_0} + \frac{2\sigma}{R_0}\left(1 - \frac{R_0}{R}\right).\end{aligned} \tag{12}$$

Plesset and Zwick (Section 1.7) start from the expression:

$$\rho_1(R\ddot{R} + \tfrac{3}{2}\dot{R}^2) = p(R) - p_0. \tag{13}$$

Hence

$$\begin{aligned}\rho_1(R\ddot{R} + \tfrac{3}{2}\dot{R}^2) &= \left\{p_v(T_R) - \frac{2\sigma}{R}\right\} \\ &\quad - \left\{p_v(T_0) - \frac{2\sigma}{R_0}\right\} = \frac{2\sigma}{R_0}\frac{T_R - T_0}{\Delta\theta_0} \\ &\quad + \frac{2\sigma}{R_0}\left(1 - \frac{R_0}{R}\right),\end{aligned}$$

i.e. equation (12) again.

1.6. *The Forster–Zuber theory*

A vapour bubble is a moving spherical heat sink, which can be found by integration of a moving instantaneous point sink [8]. The temperature of the vapour in the bubble is uniform due to its large thermal diffusivity and equals $T(R, t) = T_R(t)$ as thermodynamic equilibrium at the interface is postulated (Section 1.1).

The instantaneous superheating of the bubble boundary amounts to:

$$\begin{aligned}\Delta T_R = T_R - T &= -(T_0 - T_R) + \Delta\theta_0 \\ &= \Delta T_R^* + \Delta\theta_0.\end{aligned} \tag{14}$$

Forster and Zuber [8] and Zwick [9, 11, 12] showed that $\Delta T_R \to 0$ as time $t \to \infty$, so that the vapour temperature approximates the dew temperature (Figs. 3 and 4). This prediction is in agreement with the experimental results of Prüger (Section 1.2.2).

Forster and Zuber used the heat-conduction equation and the heat balance (cf. Carslaw and Jaeger [13]) to show that ΔT_R can be approximated by

$$\Delta T_R = \Delta\theta_0 - \frac{\rho_2 l}{\rho_1 c(\pi a)^{\frac{1}{2}}} \int_0^t \frac{R(t')\,\mathrm{d}R(t')/\mathrm{d}t'}{R(t)(t - t')^{\frac{1}{2}}}\,\mathrm{d}t'. \tag{15}$$

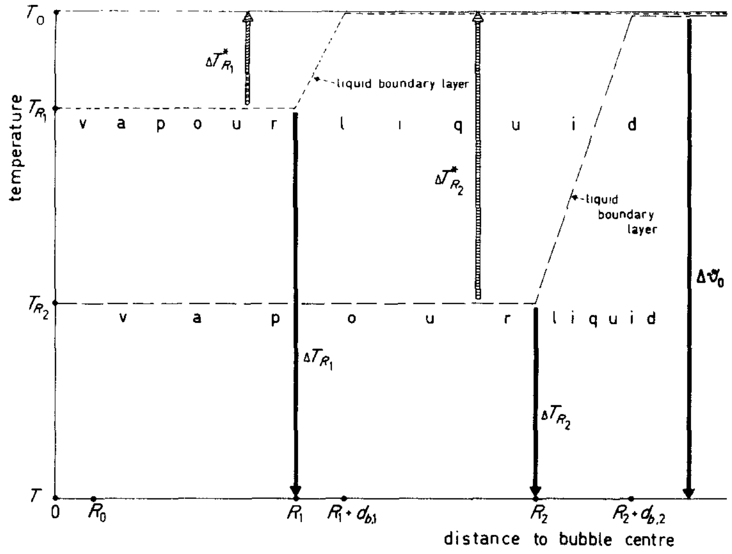

FIG. 3. *Temperature distribution* near the boundary of a growing free vapour bubble in a superheated pure liquid.

The extended Rayleigh equation is used to show that $\Delta T_R \to 0$ (Forster and Zuber) or $\Delta T^*_R \to -\Delta\theta_0$ (Plesset and Zwick) as $R \to \infty$. More exactly: $\Delta T_R \to R_0/C_1 t^{\frac{1}{2}}$, cf. equation (26)

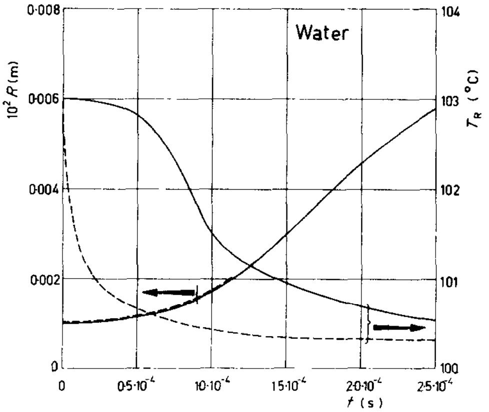

FIG. 4. *Water.* Initial growth of free vapour bubble at $\Delta\theta_0 = 3$ degC, according to Zwick (—) and to Forster and Zuber (-----).

At atmospheric pressure $R_0 = 1{\cdot}1 \times 10^{-5}$ m, and $T_R = T + \Delta T_R = 100 + 4{\cdot}5 \times 10^{-3} t^{-\frac{1}{2}}$ (cf. caption of Fig. 3). $T_R \to T$ as $t \to \infty$. The initial part of the growth curve $R \sim t^{\frac{1}{2}}$ according to Forster and Zuber's equation (23) tallies with Zwick's curve by taking a heat source of suitable strength.

A solution of equation (15)

$$R(t) = C_2 t^{\frac{1}{2}} \tag{16}$$

(where C_2 is a coefficient which depends on the liquid superheating $\Delta\theta_0$) is found by the substitution

$$t' = z_1 t. \tag{17}$$

The integral in the right-hand side of equation (15) is transformed then into one with time-independent limits:

$$\begin{aligned}\Delta T_R = T(R,t) - T &= \Delta\theta_0 - \{T_0 - T(R,t)\} \\ &= \Delta\theta_0 - \frac{\rho_2 l}{\rho_1 c(\pi a)^{\frac{1}{2}}} \dot{R}\, t^{\frac{1}{2}} \int_0^1 (1 - z_1)^{-\frac{1}{2}} \mathrm{d}z_1 \\ &= \Delta\theta_0 - \frac{\rho_2 l}{\rho_1 c(\pi a)^{\frac{1}{2}}} C_2. \end{aligned} \tag{18}$$

The following expression for the equilibrium radius (nucleus) has been used in equation (12):

$$R_0 = \frac{2\sigma}{\Delta p}\frac{\Delta T_R}{\Delta\theta_0}. \tag{19}$$

Hence $\Delta p(0) = 2\sigma/R_0$. The relation $\Delta p(t)/\Delta p(0) = \Delta T_R(t)/\Delta\theta_0$ and a more practical expression:

$$R_0 = \frac{2\sigma T}{\rho_2 l \Delta\theta_0} \tag{20}$$

are following from equation (19) in combination with the Clausius–Clapeyron equation:

$$\Delta p(t) = \frac{\rho_2 l}{T}\Delta T_R(t). \tag{21}$$

1.6.1. *Initial bubble growth.* It is seen from equation (12) in combination with the boundary conditions $R(0) = R_0$ and $\dot{R}(0) = 0$, that $\ddot{R}(0) = 0$. This means that a vapour bubble with radius R_0 initially at rest will remain in unstable equilibrium. Hence initial growth must be started by introducing a constant heat source of strength Q per unit volume and time, so that a term $Qt/\rho_1 c$ is added to the right-hand side of equation (15) and a term $(2\sigma/R_0\,\Delta\theta_0)\,Qt/\rho_1 c$ to the right-hand side of equation (12).

Initially, $\Delta T_{R_0} = \Delta\theta_0$ and the hydrodynamic equation of motion (12) is then simplified to

$$R\,\ddot{R} + \tfrac{3}{2}\dot{R}^2 = \frac{2\sigma Q}{\rho_1^2 c R_0 \Delta\theta_0}\,t. \tag{22}$$

The required particular solution is:

$$R = \left(\frac{16\sigma Q}{33\rho_1^2 c R_0 \Delta\theta_0}\right)^{\frac{1}{2}} t^{\frac{3}{2}} = \left(\frac{8\rho_2 l Q}{33\rho_1^2 c T}\right)^{\frac{1}{2}} t^{\frac{3}{2}}, \tag{23}$$

i.e. the bubble starts from critical size with a growth rate $\dot{R}$ proportional to $t^{\frac{1}{2}}$ (Fig. 4); the initial growth is independent of the liquid superheating $\Delta\theta_0$ and the equilibrium radius R_0 is proportional to $1/\Delta\theta_0$.

1.6.2. *Asymptotic bubble growth.* Inserting equations (14) and (18) into the equation of motion (12) yields:

$$\rho_1(R\,\ddot{R} + \tfrac{3}{2}\dot{R}^2) = \frac{2\sigma}{R_0\,\Delta\theta_0} \times \left\{\Delta\theta_0 - \frac{C_2\Delta\theta_0}{(\pi a)^{\frac{1}{2}} Ja}\right\} - \frac{2\sigma}{R}. \tag{24}$$

The asymptotic isobaric bubble growth is no longer governed by hydrodynamics, since both $R\ddot{R}$ and $\dot{R}^2 \sim 1/t$ and $1/R \sim 1/t^{\frac{1}{2}} \to 0$ as $t \to \infty$. Already after a few milliseconds the remaining time-independent superheating term in the right-hand side of (24) is thus balanced by the negative evaporation term. Hence:

$$R \cong \pi^{\frac{1}{2}}\, Ja(at)^{\frac{1}{2}} = C_2 t^{\frac{1}{2}} = C_1\,\Delta\theta_0 t^{\frac{1}{2}}, \tag{25}$$

i.e. the bubble growth rate $\dot{R}$ is proportional to $\Delta\theta_0$ and to $1/t^{\frac{1}{2}}$, thus decreasing continuously.

Equation (12) is reduced to: $\Delta T_R^* \to (R_0/R - 1)\,\Delta\theta_0$, which yields in combination with (14):

$$\Delta T_R \to \frac{R_0}{R}\,\Delta\theta_0 \to 0 \text{ as } R \to \infty, \quad \text{i.e. as } t \to \infty. \tag{26}$$

Hence, the vapour temperature approximates the saturation value (Fig. 4).

1.7. *The Plesset–Zwick theory*

Plesset and Zwick [9] write for the extended Rayleigh equation, cf. (10) and (12):

$$\frac{1}{2R^2\dot{R}}\frac{\mathrm{d}}{\mathrm{d}t}(R^3\dot{R}^2) = B_1\,\Delta T_R^* + \frac{2\sigma}{\rho_1 R_0}\left(1 - \frac{R_0}{R}\right). \tag{27}$$

Apparently, the coefficient

$$B_1 = \frac{2\sigma}{\rho_1 R_0}\frac{1}{\Delta\theta_0} = \frac{\rho_2 l}{\rho_1 T}. \tag{28}$$

The latent heat of vaporization is supplied to the bubble boundary by thermal conduction through a thin liquid layer adjacent to the bubble [cf. Sections 1.1 and 1.2, especially equation (1)] so that the heat inflow is:

$$\Phi_b = 4\pi R^2 k \left(\frac{\partial T}{\partial r}\right)_{r=R} = \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{4}{3}\pi\rho_2 l R^3) = 4\pi\rho_2 l R^2 \dot{R}. \tag{29}$$

The temperature at the bubble boundary (which equals the uniform vapour temperature as a consequence of the large thermal diffusivity of the vapour) is determined by the condition (29) for the temperature gradient in the liquid at the bubble boundary, in combination with the following equation, cf. equation (14):

$$\Delta T_R^* = T_R - T_0 = \Delta T_R - \Delta\theta_0 =$$

$$-\left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_0^t \frac{R^2(z_2)\left(\frac{\partial T}{\partial r}\right)_{r=R(z_2)}}{\{\int_0^t R^4(z_3)\,\mathrm{d}z_3\}^{\frac{1}{2}}}\,\mathrm{d}z_2. \qquad (30)$$

$\Delta T_R^* = 0$ if the cooling effect of evaporation is disregarded so that (27) is then simplified to Rayleigh's equation with the solution (8):

$$\dot{R} = \left(\frac{4\sigma}{3\rho_1 R_0}\right)^{\frac{1}{2}} = \text{constant.}$$

Plesset and Zwick introduce the dimensionless variables:

$$p = \left(\frac{R}{R_0}\right)^3 \text{ in place of } R \qquad (31)$$

$$u = \frac{B_2}{R_0^4}\int_0^t R^4(z_3)\,\mathrm{d}z_3 \text{ in place of } t, \qquad (32)$$

where the constant B_2 is defined as:

$$B_2 = \left(\frac{2\sigma}{\rho_1 R_0^3}\right)^{\frac{1}{2}}. \qquad (33)$$

Equation (27) is then transformed to

$$\frac{1}{6p'}\frac{\mathrm{d}}{\mathrm{d}u}\{p^{\frac{7}{3}}(p')^2\} = 1 - p^{-\frac{1}{3}}$$

$$-\kappa\int_0^u p'(z_4)\,(u - z_4)^{-\frac{1}{2}}\,\mathrm{d}z_4, \qquad (34)$$

in which $p' = \mathrm{d}p/\mathrm{d}u$ and the dimensionless parameter κ is given by

$$\kappa = \frac{B_1\rho_2 l}{3B_2 k R_0}\left(\frac{a}{\pi B_2}\right)^{\frac{1}{2}} = \left(\frac{2}{9\pi}\right)^{\frac{1}{2}}\left\{\frac{\rho_2^3 l^3\sigma^2 T}{\rho_1(\Delta\theta_0)^5 k^2 c^2}\right\}^{\frac{1}{4}}. \qquad (35)$$

The inverse transformations follow from (32) and (33):

$$R = R_0 p^{\frac{1}{3}} \qquad (36)$$

$$t = \frac{1}{B_2}\int_0^u \{p(z_4)\}^{-\frac{4}{3}}\,\mathrm{d}z_4. \qquad (37)$$

Hence

$$\dot{R} = \frac{B_2 R_0}{3}p^{\frac{2}{3}}p' \qquad (38)$$

and

$$\Delta T_R^* = \frac{B_2^2 R_0^2 \kappa}{B_1}\int_0^u p'(z_4)\,(u - z_4)^{-\frac{1}{2}}\,\mathrm{d}z_4. \qquad (39)$$

1.7.1. *Asymptotic bubble growth.* The behaviour of the bubble for $R/R_0 \gg 1$, i.e. for large t, shall now be considered. As the bubble grows, the acceleration of the growth in the left-hand side of equation (27) tends towards zero because of the cooling effect. It therefore follows from equation (34) that

$$\int_0^u p'(z_4)\,(u - z_4)^{-\frac{1}{2}}\,\mathrm{d}z_4 \to \kappa^{-1} \qquad \text{as } u \to \infty. \qquad (40)$$

The substitution

$$z_4 = \xi u \qquad (41)$$

transforms the integral in (39) into

$$u^{-\frac{1}{2}}\int_0^1 (1 - \xi)^{-\frac{1}{2}}\frac{\mathrm{d}p(\xi u)}{\mathrm{d}\xi}\,\mathrm{d}\xi \to \kappa^{-1}. \qquad (42)$$

The right-hand side of (42) is independent of time, when

$$p(u) = B_3 u^{\frac{1}{2}}, \qquad (43)$$

where B_3 is a constant which is determined by (42). The integral in the left-hand side of (42) can be expressed in the beta-function B and hence in the gamma-function Γ:

$$\int_0^1 \xi^{-\frac{1}{2}}(1 - \xi)^{-\frac{1}{2}}\,\mathrm{d}\xi = B(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1)} = \pi, \qquad (44)$$

so that $B_3 \cong 2/\pi\kappa$ and

$$p(u) \cong \frac{2}{\pi\kappa} u^{\frac{1}{2}} \qquad \text{as } u \to \infty. \tag{45}$$

Hence

$$\Delta T_R^* \to -\frac{B_2^2 R_0^2}{B_1} = -\Delta\theta_0 \qquad \text{as } t \to \infty. \tag{46}$$

This is equivalent with $\Delta T_R \to 0$ in the Forster–Zuber theory, cf. (26), i.e. the vapour temperature approximates the saturation value.

From (36) and (45) it follows that

$$R \cong R_0 \left(\frac{2}{\pi\kappa}\right)^{\frac{1}{3}} u^{\frac{1}{6}}, \tag{47}$$

and from (37) and (45) that

$$u^{\frac{1}{6}} \cong \left(\frac{B_2}{3}\right)^{\frac{1}{2}} \left(\frac{2}{\pi\kappa}\right)^{\frac{2}{3}} t^{\frac{1}{2}}. \tag{48}$$

The asymptotic approximation for the bubble radius as $t \to \infty$ follows from a substitution of (48) in (47):

$$\begin{aligned} R &\cong \left(\frac{12}{\pi}\right)^{\frac{1}{2}} \frac{k}{\rho_2 l a^{\frac{1}{2}}} \Delta\theta_0 t^{\frac{1}{2}} \\ &= \left(\frac{12}{\pi}\right)^{\frac{1}{2}} \frac{(k\rho_1 c)^{\frac{1}{2}}}{\rho_2 l} \Delta\theta_0 t^{\frac{1}{2}} = \left(\frac{12}{\pi}\right)^{\frac{1}{2}} Ja(at)^{\frac{1}{2}}. \end{aligned} \tag{49}$$

1.8. *The Scriven theory*

Scriven [14] extended the heat-conduction equation for spherical symmetry to establish the effect of radial convection resulting from unequal phase densities:

$$\frac{\partial T}{\partial t} + \varepsilon \dot{R} \frac{R^2}{r^2} \frac{\partial T}{\partial r} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r}\right). \tag{50}$$

The boundary conditions are $T(\infty, t) = T_0$ and $T(R, T) = T$ and the initial condition is $T(r, 0) = T_0$; $v(R, t) = \varepsilon\dot{R}$ and $v(r, t) = \varepsilon\dot{R}R^2/r^2$ denote the instantaneous radial velocity of liquid immediately adjacent to the bubble boundary and at greater distance from the bubble centre, respectively. Note that the expression for the velocity in equation (7) for isothermal bubble growth without mass transfer due to evaporation is in agreement with $\varepsilon = 1 - \rho_2/\rho_1 = 1$ as $\rho_2 = 0$.

Scriven assumed a solution $T(r, t) = T(s)$ on dimensional grounds, with s independent of time:

$$r = 2s(at)^{\frac{1}{2}}, \tag{51}$$

where s is independent of time, i.e. $\partial s/\partial t = 0$. At the bubble boundary we denote $s = \beta$, Hence

$$R = 2\beta(at)^{\frac{1}{2}} = C_2 t^{\frac{1}{2}}. \tag{52}$$

The arbitrary constants occurring in the two successive integrations of the resulting ordinary equation are determined by using the first boundary condition and equation (29), which is at $t = 0$ equivalent to the initial condition as $\dot{R}(0) = 0$. Differentiating $T(s)$ partially with respect to r and t yields:

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{\partial T}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial T}{\partial r}\frac{\partial r}{\partial s}$$

or

$$\frac{\partial T}{\partial t} = \frac{\mathrm{d}T}{\mathrm{d}s}\frac{\partial s}{\partial t} - \frac{\partial T}{\partial r}\frac{\partial r}{\partial s}\frac{\partial s}{\partial t} = -\frac{\partial T}{\partial r}\frac{\partial r}{\partial t}$$

with

$$\frac{\partial s}{\partial r} = \frac{s}{r}, \frac{\partial^2 s}{\partial r^2} = 0 \qquad \text{and} \qquad \frac{\partial r}{\partial t} = s\left(\frac{a}{t}\right)^{\frac{1}{2}} = 2a\frac{s^2}{r}.$$

Hence, the transformation of the partial derivatives is then given by:

$$\left.\begin{aligned} \frac{\partial T}{\partial t} &= -2a\frac{s^3}{r^2}\frac{\mathrm{d}T}{\mathrm{d}s} \\ \frac{\partial T}{\partial r} &= \frac{s}{r}\frac{\mathrm{d}T}{\mathrm{d}s} \\ \frac{\partial^2 T}{\partial r^2} &= \left(\frac{s}{r}\right)^2 \frac{\mathrm{d}^2 T}{\mathrm{d}s^2}. \end{aligned}\right\} \tag{53}$$

The partial differential equation (50) is reduced to an ordinary differential equation by substitution of (53). Since

$$\frac{\dot{R}}{a} = 2\beta\frac{s}{r}, \tag{54}$$

one finds thus:

$$\frac{\mathrm{d}^2T}{\mathrm{d}s^2} = 2(-s - s^{-1} + \varepsilon\beta^3 s^{-2})\frac{\mathrm{d}T}{\mathrm{d}s}. \quad (55)$$

This equation can be integral immediately

$$\frac{\mathrm{d}T}{\mathrm{d}s} = A_1 s^{-2} \exp(-s^2 - 2\varepsilon\beta^3 s^{-1}). \quad (56)$$

The second integration yields

$$T(s) = T_0 - A_1 \int_s^\infty z_5{}^{-2} \exp(-z_5^2 - 2\varepsilon\beta^3 z_5^{-1})\,\mathrm{d}z_5. \quad (57)$$

Application of the heat flow equation (29), and of the equations (53) and (54) yields:

$$A_1 = \frac{\rho_2 l}{\rho_1 c} 2\beta^3 \exp(\beta^2 + 2\varepsilon\beta^2). \quad (58)$$

Hence

$$T(s) = T_0 - \frac{\rho_2 l}{\rho_1 c} 2\beta^3 \exp(\beta^2 + \varepsilon\beta^2) \int_s^\infty z_5^{-2} \exp(-z_5^2 - 2\varepsilon\beta^3 z_5^{-1})\,\mathrm{d}z_5. \quad (59)$$

Equation (53) yields an expression governing bubble growth by taking $r = R$ and applying the second boundary condition:

$$Ja = 2\beta^3 \exp(\beta^2 + \varepsilon\beta^2) \int_\beta^\infty z_5^{-2} \times \exp(-z_5^2 - 2\varepsilon\beta^3 z_5^{-1})\,\mathrm{d}z_5 = \Phi(\varepsilon, \beta). \quad (60)$$

The numerical values of $\Phi(\varepsilon, \beta)$ have been computed by Scriven [14] for constant values of ε. For $\beta \to \infty$, $\Phi(\varepsilon, \infty) \cong \rho_1/\rho_2$, when $\Delta\theta_0 = l/c$; this means, that evaporation of the entire bulk of superheated liquid occurs instantaneously at this maximal superheating. This follows also from equation (70) by taking $G_h^* = 1$.

The following approximations are of practical interest for bubbles generated in nucleate boiling, cf. Fig. 5:

(i) For released bubbles (cf. Part II) at small superheatings $\Delta\theta_0$ of the bulk liquid: $\Phi(\varepsilon, 0) = 2\beta^2$, where

$$R = C_2 t^{\frac{1}{2}} = (2Ja)^{\frac{1}{2}} (at)^{\frac{1}{2}} = C_1^*(\Delta\theta_0)^{\frac{1}{2}}\, t^{\frac{1}{2}}. \quad (61)$$

(ii) For bubbles generated at the relatively large superheating θ_0 of the heating surface: $\Phi(\varepsilon, \beta) \cong (\pi/3)^{\frac{1}{2}}\beta$, provided $\rho_2 \ll \rho_1$ (at low pressures), when

$$R \cong \left(\frac{12}{\pi}\right)^{\frac{1}{2}} Ja\,(at)^{\frac{1}{2}} = C_1\theta_0 t^{\frac{1}{2}} = C_2 t^{\frac{1}{2}} \quad (62)$$

i.e. equation (49) again, with Plesset and Zwick's numerical constant (Section 1.7). The approximation (62) is more accurate than (61) for superheatings $\Delta\theta_0 \geqslant 0{\cdot}25$ degC. (Figs. 5 and 6).

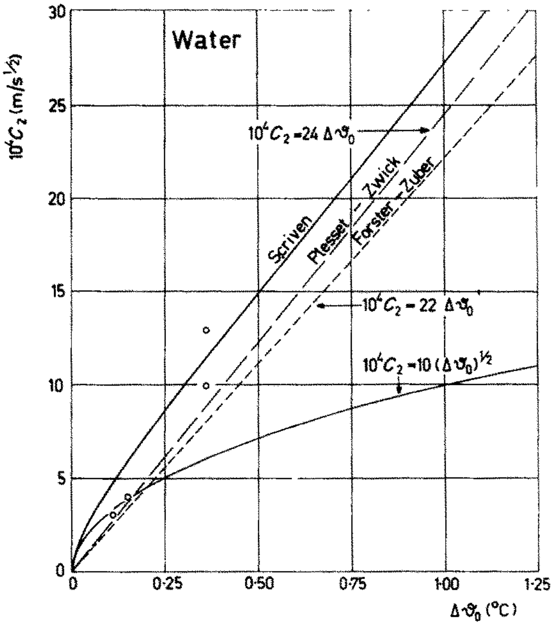

FIG. 5. *Water*. Growth factor C_2 of equation (52) for free vapour bubbles as a function of liquid superheating according to Scriven's equation (60).

The straight line $C_2 = 24 \times 10^{-4}\,\Delta\theta_0$ was predicted by Plesset and Zwick, equation (49), and is similar to Scriven's approximation for relatively large superheatings, equation (62).

The initial part of Scriven's curve can be approximated by the parabola $C_2 = 10 \times 10^{-4}\,(\Delta\theta_0)^{\frac{1}{2}}$, equation (61).

○ experimental values determined by van Stralen, cf. Part II.

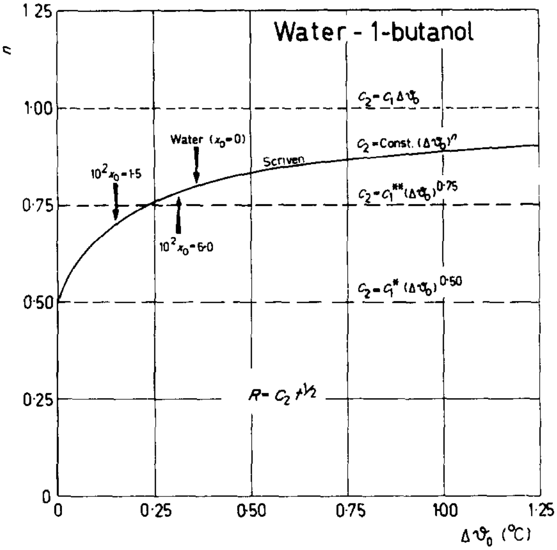

FIG. 6. *Water–1-butanol*. Exponent n in $C_2 \sim (\Delta\theta_0)^n$ as a function of superheating of bulk liquid, derived from Scriven's theory. The extreme values are $n = 0{\cdot}50$, equations (61) and (75), and $n = 1{\cdot}00$ cf. equations (62) and (74); $n = 0{\cdot}75$ is the average value in the range of superheatings occurring actually in the boiling vessel, cf. Part II.

1.9. *The Skinner–Bankoff theory*

Recently, Skinner and Bankoff [15] extended Scriven's theory by studying bubble growth in spherically symmetric temperature fields of general variation, thus including both negative superheating, i.e. subcooling (surface boiling) and initially uniform superheating. In the latter case equation (62) is reported for $R(0) = 0$. This restriction of the previous theories is reasonably allowed, because as early as $R(t)/R(0) = 7$, the deviation of $R(t)$ from equation (62) is dropped to 2 per cent. This occurs as rapidly as within 3×10^{-4} s, cf. Fig. 4.

2. BINARY MIXTURES

2.1. *Survey*

2.1.1. *Introduction.* Van Wijk, Vos and van Stralen [16–18], Scriven [14], Bruijn [19], van Stralen [20] and Skinner and Bankoff [21] extended the theory to spherically symmetric bubble growth in superheated binary mixtures. The rate of bubble growth in a pure component depends on heat flow towards the bubble boundary to satisfy the heat requirement of evaporation. In mixtures, heat diffusion is linked with mass diffusion of the more volatile component, which is rapidly exhausted in the liquid immediately adjacent to the bubble. A low concentration or mass diffusivity of the more volatile component results in a slowing down of bubble growth as the mass diffusivity is an order of magnitude smaller than the thermal diffusivity.

The theories predict the occurrence of a minimum growth rate of free bubbles (and according to van Stralen [20, 22] also of a minimum departure radius of bubbles generated at a heating surface, cf. Fig. 3 of Part I of [22], at a certain low concentration of the more volatile component. This minimum has actually been established experimentally, cf. Part II.

2.1.2. *Recent developments.* A striking phenomenon is the observed coincidence of a minimal bubble growth rate (corresponding with a maximal $\Delta T/G_d$, cf. equation (74), resulting in a minimal heat transmission to individual bubbles) and the occurrence of a maximal nucleate boiling peak flux density at the same liquid composition (Fig. 2 of Part I of [22]). Recently, this "boiling paradox" has been explained by van Stralen [22] by describing nucleate boiling as a relaxation phenomenon concerning the superheating and the excess enthalpy of the equivalent conduction layer at the heating surface. This boundary layer is pushed away periodically from the wall due to the rapid growth of succeeding bubbles on active nuclei.

The theory is based on the results of the present study of bubble growth and on the occurrence of rapid local temperature dips at the contact area between heating surface and boiling liquid, both in nucleate boiling and in film boiling with partial liquid–wall contact. The temperature fluctuations coincide with the initial generation of bubbles on nearby nuclei and have been observed by Moore and Mesler [23], and by Madsen [24]. Different interpretations of this phenomenon have created much controversy about the understanding of the mechanism of nucleate boiling. Moore and Mesler referred the fluctuations to the heating surface and as a consequence stated the hypothesis, that a very thin (order of magnitude of 1 μm) liquid microlayer between a vapour bubble and the heating surface exists and evaporates rapidly, thus being responsible for the calculated very high local heat fluxes during initial bubble growth. Contrarily, Madsen and van Stralen [22] showed the unnecessariness of this hypothesis and assumed the existence of a $l/c\theta_0$ times thicker liquid "relaxation microlayer" at the bottom of the bubble (or surrounding a part of the bubble interface, respectively), which is heated periodically during the waiting time and is cooled during bubble growth.

The theoretical background is justified by high speed Schlieren motion pictures taken by Béhar and Séméria [25] and more recently by Bähr [26].

In principle, the new theory includes the favourable effect on peak flux by any method resulting in a diminished direct vapour formation at the heating surface, which corresponds generally with an increased frequency of smaller bubbles, e.g. surface boiling, vortex flow, the use of high pressures and the application of an electrostatic field.

2.2. *The van Wijk–Vos–van Stralen theory*

Essential is the prediction of the occurrence of an increase ΔT in the dew temperature of vapour with respect to the boiling temperature of the original (bulk) liquid [16–18, 20, 22]. The minimum "effective superheating" $\Delta\tau = \Delta\theta_0 - \Delta T$ in mixtures, which occurs at a low concentration of the more volatile component, results in a maximal slowing down of bubble growth, cf. equations (4, 25, 49, 62). The original theory is shown here to be equivalent in this respect to the more recent Scriven and Bruijn theories.

A simple graphical evaluation of $\Delta T/G_d$ in dependence on liquid composition from equilibrium data follows from Fig. 7 and has been discussed in Part II of [22].

2.3. *Van Stralen's modification* [20] *of the van Wijk–Vos–van Stralen, Scriven and Bruijn theories*

2.3.1. *Analogy between heat and mass diffusion.* The equations governing the concentration of

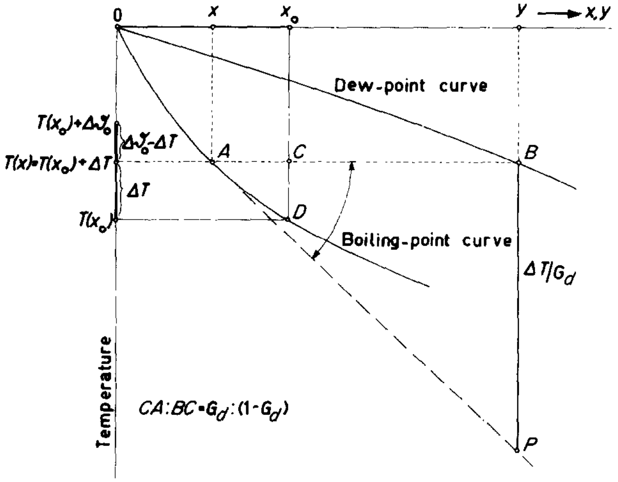

FIG. 7. *Equilibrium-diagram* for a binary system with minimum boiling point. From the material balance it follows: $1/G_d = (y - x)/(x_0 - x)$. One has for small values of G_d: $BP/CD = BA/CA$, i.e. $BP = \Delta T/G_d$, independent of G_d and of $\Delta\theta_0$.

the more volatile component are analogous to those for the temperature, which is seen by replacing T by x, a by D, $\Delta\theta_0$ by $x_0 - x$, l/c by $y - x$ and β by $\mu\beta = (a/D)^{\frac{1}{2}}\beta$, cf. equation (52). For example, equation (29) is transformed into equal rate of consumptia and flux of volatile matter into the vapour space of the bubble:

$$\rho_2(y - x)\dot{R}_m = \rho_1 D\left(\frac{\partial x}{\partial r}\right)_{r=R_m}. \tag{63}$$

It can be shown easily, that for relatively large β, $x\{R_m(t)\} \to x$, i.e. the concentration at the bubble boundary approaches a constant value; viz. the analogous equation following from equation (18) in combination with equation (52) is, if Forster and Zuber's numerical constant $\pi^{\frac{1}{2}}$ is replaced by Plesset and Zwick's and Scriven's value $(12/\pi)^{\frac{1}{2}}$:

$$\begin{aligned} x\{R_m(t)\} - x &= (x_0 - x) - [x_0 - x\{R_m(t)\}] \\ &= (x_0 - x) - 2\left(\frac{\pi}{12}\right)^{\frac{1}{2}}\frac{\rho_2}{\rho_1}(y - x)\,\mu\beta, \end{aligned} \tag{64}$$

whence, according to equation (68)

$$\begin{aligned} \frac{x\{R_m(t)\} - x}{x_0 - x} &= 1 - \left(\frac{\pi}{3}\right)^{\frac{1}{2}}\frac{\rho_2}{\rho_1}\frac{\mu\beta}{G_d} \\ &= 1 - \left(\frac{\pi}{3}\right)^{\frac{1}{2}}\frac{\mu\beta}{\Phi(\varepsilon, \mu\beta)}. \end{aligned} \tag{65}$$

It follows thus from equation (62) that for relatively large superheatings both $x\{R_m(t)\} = x$ and $y = Kx$ are independent of t.

2.3.2. *Bubble growth equation.* In binary mixtures, bubble growth is thus governed by two equations, both originating from equation (60), i.e. by the combination of a modified and an analogous expression.

One has now: $T_0 = T(x_0) + \Delta\theta_0$ and $T\{R_m(t)\} = T(x) = T(y) = T(x_0) + \Delta T$ for equilibrium evaporation (cf. Sections 1.1 and 1.6), whence equation (60) is extended to

$$Ja_0 = \frac{\rho_1 c}{\rho_2 l}(\Delta\theta_0 - \Delta T) = \frac{\rho_1 c}{\rho_2 l}\Delta\tau = \Phi(\varepsilon, \beta), \tag{66}$$

where Ja_0 is a new dimensionless group or modified Jakob number for mixtures; according to equation (69):

$$Ja_0 = \frac{C_{1,m}}{C_{1,p}}Ja = \frac{\theta_0}{(\rho_2/\rho_1)\{l/c + (a/D)^{\frac{1}{2}}\Delta T/G_d\}}. \tag{67}$$

$Ja_0 = Ja$ for pure liquids and azeotropic mixtures, as $\Delta T = 0$ then, cf. also equation (74).

The analogous equation is given by

$$\frac{\rho_1}{\rho_2}G_d = \frac{\rho_1}{\rho_2}\frac{x_0 - x}{y - x} = \Phi(\varepsilon, \mu\beta). \tag{68}$$

The second new dimensionless group G_d shall be called the vaporized mass diffusion fraction [20, 22] for individual bubbles. For relatively large β (Section 1.8) it follows from equation (66):

$$\Delta T = T(x) - T(x_0) = \left(1 - \frac{C_{1,m}}{C_{1,p}}\right)\Delta\theta_0, \tag{69}$$

and from equation (62):

$$G_d = \left(\frac{a}{D}\right)^{\frac{1}{2}}\frac{C_{1,m}}{C_{1,p}}\frac{c}{l}\Delta\theta_0 = \left(\frac{a}{D}\right)^{\frac{1}{2}}G_h^*, \tag{70}$$

where the third new dimensionless group G_h^* is called the vaporized heat diffusion fraction [20, 22]†. The ratio

$$\begin{aligned} \frac{\Delta T}{G_d}(x_0) &= \frac{\Delta\theta_0 - (\rho_2 l/\rho_1 c)\,\Phi(\varepsilon, \beta)}{(\rho_2/\rho_1)\,\Phi(\varepsilon, \mu\beta)} \\ &= \frac{\Delta\theta_0 - (\rho_2 l/\rho_1 c)\,\Phi(1 - \rho_2/\rho_1, C_{2,m}/2a^{\frac{1}{2}})}{(\rho_2/\rho_1)\,\Phi(1 - \rho_2/\rho_1, C_{2,m}/2D^{\frac{1}{2}})}, \end{aligned} \tag{71}$$

which is in the asymptotic approximaton simplified to:

$$\frac{\Delta T}{G_d} = \left(\frac{D}{a}\right)^{\frac{1}{2}}\frac{l}{c}\left(\frac{C_{1,p}}{C_{1,m}} - 1\right), \tag{72}$$

according to equations (69) and (70). $\Delta T/G_d$ is

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† Both $G_{h,m}^*$ and $G_{h,p}^*$ refer to the same equivalent conduction layer $d_{b,p} = (\pi/3)^{\frac{1}{2}}\,(at)^{\frac{1}{2}}$ as then $G_{h,p}^* = (4\pi/3)\,\rho_2 R_p^3/4\pi\rho_1 d_{b,p}R_p^2 = (2/\pi)\,(c/l)\,\Delta\theta_0$ and $G_{h,m}^* = (R_m/R_p)\,G_{h,p}^* = (C_{1.m}/C_{1,p})\,(2/\pi)\,(c/l)\,\Delta\theta_0$.

thus independent of $\Delta\theta_0$. This ratio can, according to van Stralen [17, 22] be derived by expanding ΔT in a Taylor series in $x_0 - x$ or, according to van Wijk and van Stralen [17, 18, 20, 22, 27] be evaluated graphically (in dependence on composition) from equilibrium data (Fig. 7), as

$$\frac{\Delta T}{G_d}(x_0) = -x[K(x) - 1]\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_{x=x_0} \approx -x_0\{K(x_0) - 1\}\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)_{x=x_0} \tag{73}$$

is independent of G_d and $\Delta\theta_0$, and of x for relatively small values of G_d.

The minimum value of the bubble growth constant $C_{1,m}$ in mixtures can thus easily be derived by substituting the obtained $\Delta T/G_d$ in equation (72).

The asymptotic bubble growth equation (62) is for binary mixtures extended to:

$$R_m \cong \left(\frac{12}{\pi}\right)^{\frac{1}{2}} Ja_0(at)^{\frac{1}{2}} = \left(\frac{12}{\pi}\right)^{\frac{1}{2}} \times \frac{a^{\frac{1}{2}}}{(\rho_2/\rho_1)\{l/c + (a/D)^{\frac{1}{2}}\Delta T/G_d\}}\Delta\theta_0\, t^{\frac{1}{2}} = C_{1,m}\Delta\theta_0\, t^{\frac{1}{2}} = C_{2,m}\, t^{\frac{1}{2}}. \tag{74}$$

It is shown explicitly, that a minimal $\dot{R}_m$ for free or released bubbles in mixtures is corresponding to a maximal ratio $\Delta T/G_d$, cf. also Fig. 25 of Part II, where the theoretical curves for the system water–1-butanol have been derived from the exact equation (71). The experimental values of $C_{2,m}$ for released bubbles, which have been evaluated from high-speed motion pictures, are in quantitative agreement with the theoretical predictions (Part II).

The growth equation (61) for very small superheatings is extended to (Figs. 5 and 6):

$$R_m = (2Ja_0)^{\frac{1}{2}}(at)^{\frac{1}{2}} = \left[\frac{2a}{(\rho_2/\rho_1)\{l/c + (a/D)^{\frac{1}{2}}\Delta T/G_d\}}\right]^{\frac{1}{2}}(\Delta\theta_0)^{\frac{1}{2}}\, t^{\frac{1}{2}} = C^{*}_{1,m}(\Delta\theta_0)^{\frac{1}{2}}\, t^{\frac{1}{2}}. \tag{75}$$

2.4. *The Bruijn theory*

It will be shown here, that Bruijn's theory is identical with the special case of Scriven's theory, which is obtained for $\varepsilon = 0$, i.e. in case of equal densities of vapour and liquid. This means, that the radial convection in the liquid near the bubble has been neglected and that the theory is actually only exactly valid at high pressures approximating the critical.

In spite of this deficiency, the ratio of the growth coefficients $C_{2,m}/C_{2,p}$ differs for the system water–1-butanol at atmospheric pressure only slightly from Scriven's values. Apparently, the effect of the radial convection on the concentration of the minimal bubble growth rate in binary systems is only slight. Bruijn's temperature (and analogously for mass diffusion) equation is:

$$T(s) = T_0 - \frac{A_2}{2s}e^{-s^2} + A_2\int_s^\infty \exp(-z_5^2)\,\mathrm{d}z_5, \tag{76}$$

where A_2 is a constant of integration, the value of which is determined by taking $s = \beta$:

$$A_2 = \frac{\rho_2 l}{\rho_1 c}4\beta^2\exp\beta^2. \tag{77}$$

Hence

$$\Delta\theta_0 = T_0 - T = \frac{\rho_2 l}{\rho_1 c}4\beta^3\exp\beta^2\left\{\frac{\exp(-\beta^2)}{2\beta} - \int_\beta^\infty \exp(-z_5^2)\,\mathrm{d}z_5\right\} = \frac{\rho_2 l}{\rho_1 c}\left\{2\beta^2 - 4\beta^3\exp\beta^2\int_\beta^\infty \exp(-z_5^2)\,\mathrm{d}z_5\right\}. \tag{78}$$

One has, on the other hand by taking $\varepsilon = 0$ in equation (60):

$$\Delta\theta_0 = \frac{\rho_2 l}{\rho_1 c}\left[2\beta^3\exp\beta^2\int_\beta^\infty z_5^2\exp(-z_5^2)\,\mathrm{d}z_5\right]. \tag{79}$$

Partial integration of the integral in the right-hand side of equation (79) yields:

$$\int_{\beta}^{\infty} z_5^2 \exp(-z_5^2)\,\mathrm{d}z_5 = \frac{1}{\beta}\exp(-\beta^2) - 2\int_{\beta}^{\infty} \times \exp(-z_5^2)\,\mathrm{d}z_5. \qquad (80)$$

Finally, substitution of (80) in (79) gives (78).

2.5. *The Skinner–Bankoff theory*

Skinner and Bankoff [21, 15] extended Scriven's theory for initially uniformly superheated and homogeneous binary mixtures to arbitrary spherically symmetric initial conditions. Recently, Yatabe and Westwater [28] have shown, that the numerical solutions for ethylene-glycol–water (a non-azeotropic system, where water is the more volatile component) are almost identical to those of Scriven. This is apparently due to the property of the mass diffusion equation, which is analogous to the heat diffusion equation (50), that the concentration gradient at the bubble boundary and thus $\dot{R}_m$, cf. equation (63), are mainly governed by the nearby concentrations.

2.6. *The van Stralen extension to dilute salt solutions*

An elevation of boiling point occurs in binary systems with a nonvolatile component, which is proportional to its concentration; e.g. an accumulation of salt can appear in the liquid layer adjacent to the boundary of a steam bubble; the corresponding boiling point is

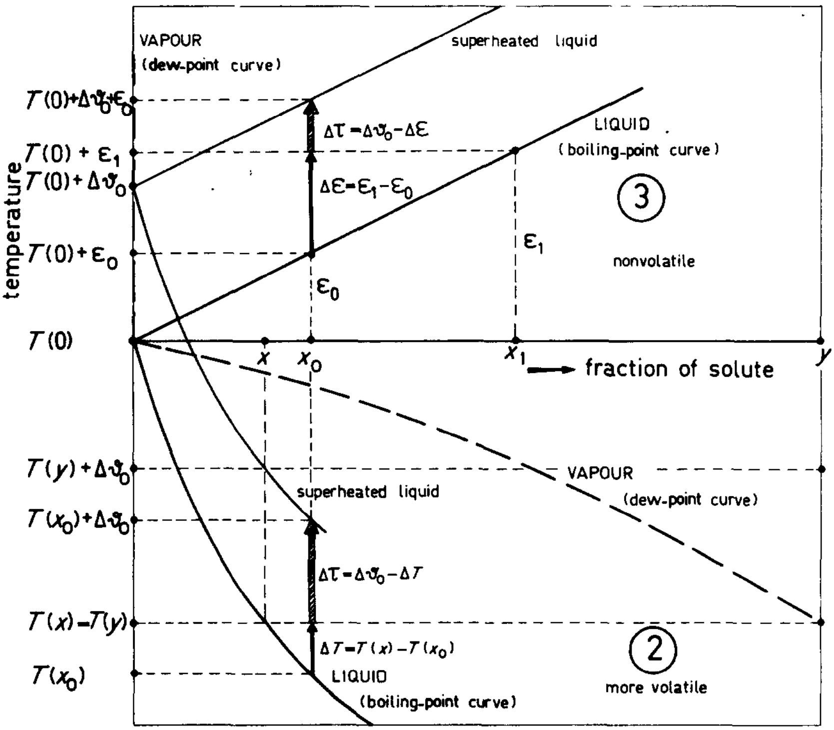

FIG. 8. *Temperature-composition* diagrams for binary systems with a more volatile component, 2, cf. Fig. 7, and a nonvolatile component, 3, e.g. dilute salt solution.

In the latter case $K = y/x = 0$, and the dew point of vapour or the boiling point of liquid at the bubble boundary is increased with an amount ε_1, the boiling point of bulk liquid with $\varepsilon_0 < \varepsilon_1$; $\Delta\varepsilon = \varepsilon_1 - \varepsilon_0$.

Bubble growth is slowed down as the effective superheating $\Delta\tau = \Delta\theta_0 - \Delta\varepsilon < \Delta\theta_0$.

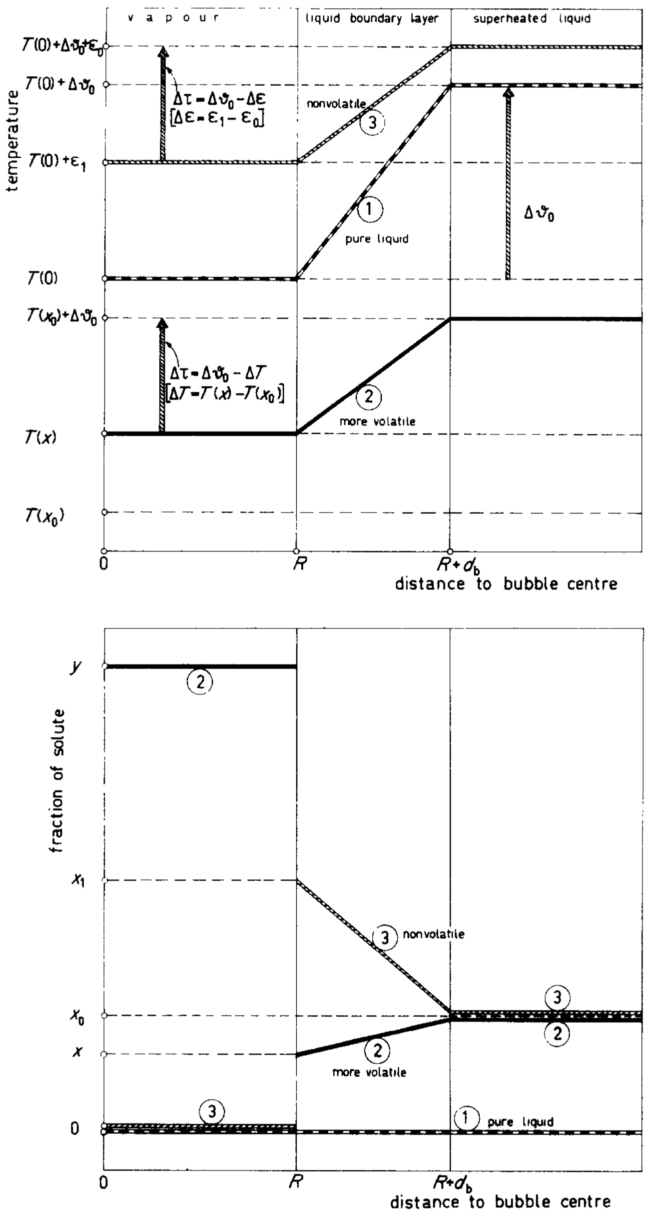

FIG. 9. *Temperature* and *composition* as a function of distance to bubble centre (cf. Figs. 7 and 8). R = radius of free vapour bubble, d = thickness of liquid boundary layer.

① Pure, less volatile component; ② binary system with more volatile component; ③ binary system with nonvolatile component.

increased by an amount ε_1, which exceeds the elevation ε_0 in the bulk liquid with an amount (Fig. 8):

$$\Delta\varepsilon = \varepsilon_1 - \varepsilon_0. \tag{81}$$

The "effective superheating" (Section 2.2), i.e. the difference in temperature between the superheated bulk liquid and the dew point of vapour equals now:

$$\Delta\tau = (T + \Delta\theta_0 + \varepsilon_0) - (T + \varepsilon_1) = \Delta\theta_0 - \Delta\varepsilon. \tag{82}$$

Consequently, a slowing down of bubble growth rates is also predicted in this case. The occurrence of a decreasing bubble size in coincidence with an increasing peak flux density in nucleate boiling has actually been observed by Samuel [29]. It follows by taking $y = 0$ in equation (68), that $0 < G_d = 1 - x_0/x$, where $x > x_0$ in this case, i.e. the mass diffusion of the nonvolatile component has been reversed in comparison with the behaviour of a more volatile component, and is directed now from the bubble boundary away to the surrounding liquid (removal of salt instead of supply towards the interface).

The asymptotic bubble growth equation (74) is now replaced by:

$$R_m \cong \left(\frac{12}{\pi}\right)^{\frac{1}{2}} \frac{a^{\frac{1}{2}}}{(\rho_2/\rho_1)\,\{l/c + (a/D)^{\frac{1}{2}}\,\Delta\varepsilon/G_d\}}\,\Delta\theta_0\, t^{\frac{1}{2}} = C_{1,m}\,\Delta\theta_0\, t^{\frac{1}{2}}. \tag{83}$$

In Fig. 9, the temperature and mass fraction vs. distance to the bubble centre diagram is shown both for binary systems with a more volatile and a nonvolatile component.

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Résumé—On passe en revue les théories concernant la croissance à symétrie sphérique de bulles libres dans des liquides purs surchauffés initialement de façon uniforme. La base physique du modèle de Bošnjaković est exposée à partir des expériences de Jakob et Fritz, donnant la distribution de température dans les liquides en ébullition, et par Heydrich et Prüger, montrant que l'équilibre thermodynamique existe à l'interface vapeur-liquide pendant l'évaporation en régime permanent de liquides surchauffés sans ébullition.

En accord avec Forster et Zuber, et avec Plesset et Zwick, la croissance des bulles dans un liquide par surchauffé suivant l'équation dynamique de Rayleigh du mouvement isotherme [$R \cong (2\Delta\rho/3\rho_1)^{\frac{1}{2}}t$ pour une cavité sphérique qui se dilate avec une pression différentielle constante Δp] est ralentie par la diffusion de la chaleur vers la frontière de la bulle pour satisfaire la condition de chaleur latente de l'évaporation. Pour une croissance asymptotique ($R_p \cong c_{1,p}\Delta\theta_0 t^{\frac{1}{2}}$), la dynamique des bulles et l'influence de la viscosité sont négligeables puisque $\Delta p \rightarrow 0$ lorsque $t \rightarrow \infty$ (croissance isobare). L'équilibre thermodynamique à la frontière de la bulle obéit à l'équation généralisée de Rayleigh en accord avec les résultats de Prüger.

Dans les mélanges binaires surchauffés, la croissance de la bulle est diminuée davantage à cause de la diffusion massique analogue du composant le plus volatil en accord avec van Wijk, Vos et van Stralen, avec Scriven, Bruijn, van Stralen et Skinner et avec Bankoff. La modification de van Stralen montre l'équivalence théorique des diverses théories. La température du point de rosée de la vapeur est augmentée d'une quantité ΔT par rapport à la température d'ébullition du liquide original. En conséquence, l'existence d'une vitesse minimale de croissance de bulle (correspondant) une valeur maximale $\Delta T/G_d$) est prévue pour une certaine concentration faible, du composant plus volatif. Ceci conduit au "paradoxe de l'ébullition", qui peut être expliqué par la théorie de la "microcouche de relaxation" de van Stralen.

Zusammenfassung—Es wird ein Überblick über die Theorien gegeben, die das kugelsymmetrische Wachstum von freien Blasen in anfangs gleichmässig überhitzten reinen Flüssigkeiten betreffen. Die physikalische Grundlage des Bošnjaković-Modells wird auf Grund von Versuchen von Jakob und Fritz, die die Temperaturverteilung in siedenden Flüssigkeiten angeben, und von Heidrich und Prüger, die zeigen, dass bei stationärer Verdampfung überhitzter Flüssigkeiten ohne Blasenbildung an der Dampf-Flüssigkeitsgrenzfläche thermodynamisches Gleichgewicht herrscht, entwickelt.

Nach Forster und Zuber, sowie Plesset und Zwick wird das Blasenwachstum in einer überhitzten reinen Flüssigkeit, wie es sich aus der Rayleigh'schen Bewegungsleichung für isotherme Bewegung ergibt [$R \cong (2\Delta p/3\rho_1)^{\frac{1}{2}}t$ für eine wachsende kugelförmige Blase mit konstantem Überdruck Δp] durch den Wärmetransport zur Blasengrenzfläche, der zur Deckung des Bedarfs an latenter Verdampfungswärme notwendig ist, verlangsamt. Für asymptotisches Wachstum ($R_p \cong C_{1,p}\Delta_0 t^{\frac{1}{2}}$) sind Blasendynamik und Einfluss von Zähigkeit und Oberflächenspannung vernachlässigbar, da $\Delta p \rightarrow 0$ bei $t \rightarrow \infty$ (isobares Wachstum). Thermodynamisches Gleichgewicht an der Blasengrenzfläche folgt aus der erweiterten Rayleigh-Gleichung in Übereinstimmung mit Prüger's Ergebnissen.

In überhitzten binären Gemischen wird das Blasenwachstum nach van Wijk, Vos und van Stralen, Scriven, Bruijin, van Stralen und Skinner und Bankoff wegen der analogen Diffusion der flüchtigeren Komponente weiter verzögert. van Stralen's Modifikation zeigt die physikalische Äquivalenz der verschiedenen Theorien. Die Taupunktstemperatur des Dampfes wird um den Betrag ΔT gegenüber der Siedetemperatur der ursprünglichen Flüssigkeit erhöht. Als Folge davon wird das Auftreten einer minimalen Wachstumsgeschwindigkeit (entsprechend einem maximalen $\Delta T/G_d$) bei einer bestimmten geringen Konzentration der flüchtigeren Komponente, vorausgesagt. Das führt zum "Siede-Paradoxon", das durch van Stralen's "Relaxationsmikroschicht"-Theorie erklart werden kann.

Аннотация—Дается обзор теоретических работ о сферически симметричном росте свободных пузырьков в первоначально равномерно перегретой чистой жидкости. В основу физической модели Бошняковича положены эксперименты Якоба и Фритца по распределению температуры кипящей жидкости, а также эксперименты Гейдриха и Прюгера, которые показывают, что на границе раздела фаз пар–жидкость во время стационарного испарения перегретой жидкостью без кипения существует термодинамическое равновесие.

Согласно Фостеру и Зуберу, а также Плессе и Цвику, рост пузырьков в перегретой чистой жидкости, как следует из динамического уравнения изотермического движения Релея [$R \cong (2\Delta/3\rho_1)^{\frac{1}{2}}t$ для расширяющейся сферической полости с постоянным избыточным давлением Δp], замедляется диффузией тепла, направленной к границе пузырьков, для того чтобы удовлетворить требованию скрытой теплоты парообразования. При асимптотическом росте ($R_p \cong C_{1,p}\Delta\theta_0 t^{\frac{1}{2}}$) динамикой пузырьков и влиянием вязкости и поверхностного натяжения можно пренебречь, т.к. $\Delta p \to 0$ при $t \to \infty$ (изобарный рост). Из обобщенного уравнения Релея в соответствии с данными Прюгера следует, что на поверхности пузырьков имеет место термодинамическое равновесие.

Согласно работам Ван-Вийка, Воса и Ван-Штралена, Брюийна, Ван-Штралена и Скиннера, а также Банкоффа, в перегретых бинарных смесях рост пузырьков ослабляется также благодаря диффузии массы более летучих компонентов. Модификация Ван-Штралена указывает на физическое подобие различных теорий. Точка росы для пара увеличивается на величину ΔT по отношению к температуре кипения первоначальной жидкости. В результате минимальную скорость роста пузырьков (соответствующую максимальному отношению $\Delta T/G_d$) можно рассчитать при определенной низкой концентрации более летучего компонента. Это приводит к «парадоксу кипения», который можно объяснить согласно теории Ван-Штралена о «релаксации микрослоя».